1 Intro

Thm: $(\mathbb{C}, +, \cdot)$ is a field.

Lemma: (Binomial formula) If $z_1, z_2 \in \mathbb{C}, n \in \mathbb{N}$, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}.$$

Def: Distance between z_1 and z_2.

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Lemma: Triangle Inequality

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

Def. Complex Conjugate of $z = x + iy$ is $\overline{z} = x - iy$
Properties:

1. $\overline{\overline{z}} = z$
2. $|\overline{z}| = |z|$
3. $\overline{z} = z \iff z \in \mathbb{R}$
4. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
5. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, $\overline{\frac{z_1}{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}$
6. $\text{Re}\{z\} = \frac{1}{2}(z + \overline{z})$, $\text{Im}\{z\} = \frac{1}{2i}(z - \overline{z})$
7. $|z|^2 = z\overline{z} = \overline{z}z$
8. $|z_1 z_2| = |z_1||z_2|$, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Def. $\arg z$ is the set of all arguments. $\text{Arg}\, z$ is the principal argument i.e. $\text{Arg}\, z \in (-\pi, \pi]$. $\arg z = \{\text{Arg}\, z + 2k\pi | k \in \mathbb{Z}\}$
Properties:

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
2. $\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$

2 Complex Roots

Solutions of $z^n = z_0 = r_0 e^{i\theta_0}$ are $\omega_k = \sqrt[n]{r_0} e^{i\frac{\theta_0 + 2k\pi}{n}}$ for $k = 0, 1, \cdots, n-1$. Principal root is ω_0.

3 Topology

Def. $B_\epsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \epsilon\}$
$B'_\epsilon(z_0) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < \epsilon\}$

Def. z is an interior point of a set S, if $\exists \epsilon : B_\epsilon(z) \subset S$
Def. z is an exterior point of a set S, if $\exists \epsilon : S \cap B_\epsilon(z) = \emptyset$
Def. z is a boundary point of a set S, if $\forall \epsilon : S \cap B_\epsilon(z) \neq \emptyset \wedge S^c \cap B_\epsilon(z) \neq \emptyset$

Def. S is open if it contains no boundary points.
Def. S is closed if it contains all boundary points.
Def. An open set $S \subseteq \mathbb{C}$ is called **connected** iff each pair of points in S can be joined by a polygonal line.
Def. S is a domain if S is open and connected.
Def. S is a region if S is a domain but with some boundary points.
Def. S is bounded if $\exists R > 0 : S \subset B_R(0)$.
Def. accumpulation points / limit points, z_0 is called an accumulation point of a set $S \subseteq \mathbb{C}$ if $\forall \epsilon : B'_\epsilon(z_0) \cap S \neq \emptyset$ (i.e. there is a convergent sequence to z_0 whose entries are in S)

4 Limits

Def. $\lim_{z\to z_0} f(z) = w_0$ if $\forall \epsilon > 0, \exists \delta > 0 : 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$. Alternatively, $\lim_{z\to z_0} f(z) = w_0$ if $\forall \epsilon > 0, \exists \delta > 0 : f(B'_\delta(z_0)) \subset B_\epsilon(w_0)$.

Thm. If $\lim_{z\to z_0} f(z)$ exists then it is unique.

Properties of Limit

1. $\lim_{z\to z_0} f(z) = w_0 \iff \lim_{z\to z_0} \operatorname{Re} f(z) = \operatorname{Re} w_0 \wedge \lim_{z\to z_0} \operatorname{Im} f(z) = \operatorname{Im} w_0$
2. If $\lim_{z\to z_0} f(z) = w_1$ and $\lim_{z\to z_0} g(z) = w_2$
   a) $\lim_{z\to z_0} af(z) + bg(z) = aw_1 + bw_2$
   b) $\lim_{z\to z_0} f(z)g(z) = w_1 w_2$
   c) If $w_2 \neq 0$, $\lim_{z\to z_0} f(z)/g(z) = w_1/w_2$
3. For a polynomial $P(\cdot)$, $\lim_{z\to z_0} P(z) = P(z_0)$.

Def. Neighborhood of ∞, $B_R(\infty) := \{z \in \mathbb{C} || z| > R\}$

Def. $\lim_{z\to z_0} f(z) = \infty$ if $\forall R > 0, \exists \delta > 0 : z \in B'_\delta(z_0) \implies f(z) \in B_R(\infty)$
i.e. $\forall R > 0, \exists \delta > 0 : 0 < |z - z_0| < \delta \implies |f(z)| > R$
Def. $\lim_{z\to\infty} f(z) = w_0$ if $\forall \epsilon > 0, \exists R > 0 : z \in B_R(\infty) \implies f(z) \in B_\epsilon(w_0)$
i.e. $\forall \epsilon > 0, \exists R > 0 : |z| > R \implies |f(z) - w_0| < \epsilon$
Def. $\lim_{z\to\infty} f(z) = \infty$ if $\forall R > 0, \exists r > 0 : z \in B_r(\infty) \implies f(z) \in B_R(\infty)$
i.e. $\forall R > 0, \exists r > 0 : |z| > r \implies |f(z) - w_0| > R$

Thm

1. $\lim_{z\to z_0} f(z) = \infty$ if $\lim_{z\to z_0} \frac{1}{f(z)} = 0$
2. $\lim_{z\to\infty} f(z) = w_0$ if $\lim_{z\to 0} f(z^{-1}) = w_0$
3. $\lim_{z\to\infty} f(z) = \infty$ if $\lim_{z\to 0} \frac{1}{f(z^{-1})} = 0$

5 Continuity

Def. f is continuous (CTS) at z_0 if (1) $f(z_0)$ is defined, (2) $\lim_{z\to z_0} f(z)$ exists, (3) $\lim_{z\to z_0} f(z) = f(z_0)$. i.e. $\forall \epsilon > 0, \exists \delta > 0 : |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$.

Thm1. $f : A \to B, g : B \to C \quad A, B, C \subset \mathbb{C}$. If f is CTS at z_0 and g is CTS at $f(z_0)$, then $g \circ f : A \to C$ is CTS at z_0.

Thm2. If f is CTS at z_0, $f(z_0) \neq 0$, then $f \neq 0$ in a whole neighborhood of z_0.

Thm3. $f(z) = u(x, y) + iv(x, y)$, f is CTS at $z_0 = x_0 + iy_0$ if and only if u, v are CTS at (x_0, y_0).

Thm4. If $f : R \to \mathbb{C}$ is CTS in a closed bounded region R, there exists a real number $M > 0$ such that

$$\forall z \in R : |f(z)| \leq M$$

but with equality for at least one $z_0 \in R$

6 Derivatives

Def. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Rmk: $\overline{z}$, $\operatorname{Re} z$, $\operatorname{Im} z$ are not differentiable anywhere.

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1. $\frac{\mathrm{d}}{\mathrm{dz}} c = 0$
2. $\frac{\mathrm{d}}{\mathrm{dz}} z = 1$
3. $\frac{\mathrm{d}}{\mathrm{dz}}(cf(z)) = c\frac{\mathrm{d}}{\mathrm{dz}}f(z)$
4. $\frac{\mathrm{d}}{\mathrm{dz}} z^n = nz^{n-1}$
5. $(f+g)' = f' + g'$
6. $(fg)' = f'g + fg'$
7. $(f \circ g)' = (f' \circ g)g'$
8. When $g(z) \neq 0$, $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

Cauchy-Riemann Equations (Necessary Condition)
Thm. If $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy$ is differentiable at z_0, then the partial derivatives of u and v exist and satisfy certain equations:

$$u_x = v_y \qquad u_y = -v_x$$

Further, $f'(z_0) = u_x + iv_x$.

Thm. Let $f(z) = u(x,y) + iv(x,y)$ be defined throughout some ϵ-neighborhood of $z_0 = x_0 + iy_0$ and suppose that
(a) u_x, u_y, v_x, v_y exist everywhere in the neighborhood.
(a) these partials are CTS at (x_0, y_0) and satisfy C-R.
Then $f'(z_0)$ exists and its value is $f'(z_0) = (u_x + iv_x)(x_0, y_0)$.

C-R Polar form.
Let $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ then if f is differentiable at z_0 then

$$ru_r = v_\theta \quad u_\theta = -rv_r.$$

Thm. C-R Sufficient (Polar form).
Let $f(z) = u(r,\theta) + iv(r,\theta)$ be defined in some ϵ-neighborhood of a nonzero point $z_0 = r_0, e^{i\theta_0}$ and suppose that
(a) $u_r, u_\theta, v_r, v_\theta$ exist everywhere in the neighborhood
(b) and CTS at (r_0, θ_0) and satisfy polar C-R (i.e. $ru_r = v_\theta, u_\theta = -rv_r$) at (r_0, θ_0).
Then $f'(z_0)$ exists and $f'(z_0) = e^{-i\theta}(u_r + iv_r)$

7 Special Types of Functions

7.1 Analytic Functions

Def. Analytic functions a.k.a. Holomorphic functions.
1. f is analytic at a point z_0 if it is analytic in some neighborhood of z_0.
2. Consider S an open set, $f : S \to \mathbb{C}$ is analytic in S, if $\forall z \in S : f'(z)$ exists.
3. f is entire if $f : \mathbb{C} \to \mathbb{C}$ is analytic in $\mathbb{C}$.

Properties of analytic functions.

1. f, g analytic in S then $f + g$, fg, and $\frac{f}{g}$ if $g \neq 0$ in S are analytic.
2. $g \circ f$ chain rule holds.
3. f analytic in a domain D implies f is CTS in D and C-R Eqs are satisfied in D.
4. If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D. (Proved in lecture W4A)

Def. z_0 is called a singular point if f is not analytic at z_0 but is analytic at some point in every neighborhood.

7.2 Harmonic Functions

Def. For $D \subseteq \mathbb{R}^2$ $H : D \to R$ is harmonic if (1) H has CTS partial derivatives up to 2nd order and satisfies Laplace's Equation,

$$H_{xx} + H_{yy} = 0$$

Thm. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its components u and v are harmonic.

7.3 Elementary Functions

Def. $e^z = e^x(\cos y + i \sin y)$.

Properties

1. $(e^z)' = e^z$. e^z is entire.
2. $|e^z| = e^x$. $\arg e^z = y + 2n\pi$ for all $n \in \mathbb{Z}$.
3. e^z is periodic with period $2\pi i$.
4. $\forall z \in \mathbb{C} : e^z \neq 0$
5. $e^{z_1} e^{z_2} = e^{z_1 + z_2}$, $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$
6. $e^0 = 1$, $\frac{1}{e^z} = e^{-z}$

Def. For $z = re^{i\theta} \neq 0$, $\log z = \ln r + i(\theta + 2n\pi)$ for $n \in \mathbb{Z}$
$\operatorname{Log} z = \ln r + i \operatorname{Arg} z$
$\log z = \operatorname{Log} z + i2n\pi$ for $n \in \mathbb{Z}$.

Log is not CTS on $\mathbb{C} \setminus \{0\}$, accordingly Branches fix it.
Fix $\alpha \in \mathbb{R}$ restrict value of $\theta \in \arg z$ to $\alpha < \theta < \alpha + 2\pi$.
The define $\log z = \ln r + i\theta$ on $r > 0, \alpha < \theta < \alpha + 2\pi$ and we have CTS and analytic on its domain.

Def. A branch of a multi-valued function f is any single-valued function F that is analytic in some domain D and for which $F(z)$ has one of the values of $f(z)$.
Def. A branch cut is a line or curve that is introduced to define a branch.
Def. Branch points are points on the branch cut that are singular points or points that are shared by all branch cuts.

Identities of log

1. $\log(z_1 z_2) = \log z_1 + \log z_2$
2. $\log(z_1/z_2) = \log z_1 - \log z_2$
3. for $n \in \mathbb{Z}, z \neq 0$ $z^n = e^{n \log z}$
4. for $n \in \mathbb{Z} \setminus \{0\}$ $z^{1/n} = e^{\frac{1}{n} \log z}$

Def. Power functions:
Fix $c \in \mathbb{C}$.
$z^c = e^{c \log z}$ (multi-valued)
Branch cuts are teh same as logarithm.
On a branch cut of z, $\frac{\mathrm{d}}{\mathrm{d}z} z^c = cz^{c-1}$.

Def. $c^z = e^{z \log c}$, specify a value of $\log c$ to make the function single-valued and entire.

$$\frac{\mathrm{d}}{\mathrm{d}z} c^z = c^z \log c$$

8 Trigonometric Functions

Def.

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) := \frac{e^{iz} + e^{-iz}}{2} \qquad z \in \mathbb{C}$$

and

$$\sinh(z) := \frac{e^{z} - e^{-z}}{2} \qquad \cosh(z) := \frac{e^{z} + e^{-z}}{2} \qquad z \in \mathbb{C}$$

Properties

1. $\sin(z), \cos(z)$ are entire (usual derivatives)
2. sin is odd, cos is even.
3. $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$ and $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$
4. $\sin^2(z) + \cos^2(z) = 1$
5. $\sin z = \sin x \cosh y + i \cos x \sinh y$, $\cos z = \cos x \cosh y - i \sin x \sinh y$
6. $|\sin z|^2 = \sin^2 x + \sinh^2 y$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$

Def. A zero of $f(z)$ is a $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$

Thm. The zeros of $\sin z$ and $\cos z$ in $\mathbb{C}$ are the same as $\sin x$ and $\cos x$ in $\mathbb{R}$.
i.e. $\sin z = 0 \iff z \in \pi\mathbb{Z}$ and $\cos z = 0 \iff z \in \frac{\pi}{2} + \pi\mathbb{Z}$

Properties of Hyperbolic Functions

1. $(\sinh z)' = \cosh z$, $(\cosh z)' = \sinh z$
2. $\cosh^2(z) = 1 + \sinh^2(z)$
3. $\sinh(iz) = i \sin z$, $\cosh(iz) = \cos z$
4. Thm: $\sinh z = 0 \iff z \in \pi i\mathbb{Z}$, and $\cosh z = 0 \iff z \in \frac{\pi}{2} i + \pi i\mathbb{Z}$

Def. A function is conformal if it preserves angles locally.
i.e.
An analytic complex-valued function is conformal at z_0 if whenever r_1, r_2 are smooth curves passing through z_0 at $t = 0$ with nonzero tangents, then the curves $f \circ r_1, f \circ r_2$ have non-zero tangents at $f(z_0)$ and the angle from $r_1'(0)$ to $r_2'(0)$ and the angle from $(f \circ r_1)'(0)$ to $(f \circ r_2)'(0)$ are the same.

A conformal mapping $f : D \to V$ (with D, V domains) is a bijective analytic function that is conformal at each point of D.
If such an f exists we say D and V are conformally equivalent.
Alt def:
f is conformal in D if F is analytic in D and $\forall z \in D : f'(z) \neq 0$.

If z_0 is a critical point of $f(z)$, there is an integer $m \geq 2$ (specifically the smallest integer $f^{(m)}(z_0) \neq 0$) such that the angle between two smooth curves passing through z_0 is multiplyied by m under f.

If $f(z)$ is conformal at z_0, it has a local inverse there. That is $w_0 = f(z_0)$, $\exists!$ function such that $z = g(w)$ is defined and analytic in a neighborhood of w_0 denoted as N such that $g(w_0) = z_0$ and $f(g(w)) = w$ for all $w \in N$.
Further $g'(w) = \frac{1}{f'(z)}$.

9 Integrals

Def. A path $w : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ such that $w(t) = u(t) + iv(t)$.
Def. $w'(t) = u'(t) + iv'(t)$ if u' and v' exist at t.
Properties:

- If $f : \mathbb{C} \to \mathbb{C}$ analytic, u, v differentiable at a point $t \in \mathbb{R}$, then $\frac{\mathrm{d}}{\mathrm{d}t} f(w(t)) = f'(w(t))w'(t)$
- Mean Value Theorem Does NOT Hold

Def. Integral of $w(t)$.

$$\int_a^b w(t)\,\mathrm{d}t := \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t$$

- $\mathrm{Re}\left\{\int_a^b w(t)\,\mathrm{d}t\right\} = \int_a^b \mathrm{Re}\{w(t)\}\,\mathrm{d}t$, and $\mathrm{Im}\left\{\int_a^b w(t)\,\mathrm{d}t\right\} = \int_a^b \mathrm{Im}\{w(t)\}\,\mathrm{d}t$
- Fund. Thm. of Calc. If $W'(t) = w(t)$ then $\int_a^b w(t)\,\mathrm{d}t = W(b) - W(a)$
- $\left|\int_a^b w(t)\,\mathrm{d}t\right| \leq \int_a^b |w(t)|\,\mathrm{d}t$

9.1 Contours

Defs $x(t), y(t) : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$

$$C : z(t) = x(t) + iy(t) \qquad \forall t \in [a, b]$$

C is an **arc** if x, y are CTS
An arc C is a **simple arc (Jordan arc)** if it does not cross itself.

An arc C is a **simple closed arc (Jordan curve)** if it is simple except for the fact that $z(b) = z(a)$
For closed curves, we call counterclock-wise **positively oriented**.
If x, y are differentiable on $[a, b]$, and x', y' is CTS on $[a, b]$, then we call C a **differentiable arc**.
A **smooth arc** is a differentiable arc C such that $\forall t \in (a, b) : z'(t) \neq 0$.
A smooth arc has unit tangent vector $T = \frac{z'(t)}{|z'(t)|}$ and arc length $L = \int_a^b |z'(t)| \, \mathrm{d}t$.
A **Contour** is a piecewise smooth arc. (consists of a finite number of smooth arcs joined end-to-end.)
simple closed contour is a contour that is also a simple closed arc.
Rmk: Parametrizations of arcs are not unique.

Def:
Let $f : \mathbb{C} \to \mathbb{C}$ and C a contour parametrized by $z(t) = x(t) + iy(t)$ with $t \in [a, b]$ with $f = u + iv$ P.W. CTS along C.

$$\int_C f(z) \, \mathrm{d}z := \int_a^b f(z(t)) z'(t) \, \mathrm{d}t = \int_C (u + iv)(\mathrm{d}x + i\mathrm{d}y)$$

Properties

- For $-C$ with parameterization $z(-t)$ for $t \in [-b, -a]$, then $\int_{-C} f(z)\mathrm{d}z = -\int_C f(z)\mathrm{d}z$.
- If C_1 ends at the point where C_2 begins $C_1 + C_2$ is their joining and $\int_{C_1+C_2} f(z) \, \mathrm{d}z = \int_{C_1} f(z) \, \mathrm{d}z + \int_{C_2} f(z) \, \mathrm{d}z$
- $\int_C f(z) \, \mathrm{d}z$ is independent of the parameterization of C.

Thm: ML Estimate
C contour of length L, f is p.w. CTS on C.

$$(\exists M \geq 0 : \forall z \in C : |f(z)| \leq M) \implies \left| \int_C f(z) \, \mathrm{d}z \right| \leq ML$$

Rmk: The contour integral depends on teh contour (not just its end points).

Def: F on D is an anti-derivative of CTS $f : D \to \mathbb{C}$ on D if $F' = f$ on D.
Properties:

- F is analytic on D.
- Anti-derivatives differ up to a constant on D

Thm (Fundamental Theorem of Contour Integrals)
Suppose f is CTS in D; Then the following statements are equivalent:

1. $f(z)$ has an anti-derivative $F(s)$ throughout D
2. integrals of $f(z)$ along contours lying entirely in D extending from any fixed point z_1 to any fixed point z_2 have the same value. Then for contours C_1, C_2 with shared endpoints z_1, z_2, $\int_{C_1} f(z)\ \mathrm{d}z = \int_{C_2} f(z)\ \mathrm{d}z := \int_{z_1}^{z_2} f(z)\ \mathrm{d}z = F(z_2) - F(z_1)$
3. $\oint_C f(z)\ \mathrm{d}z = 0$ for all closed contours in C in D.

Def. A **simply connected domain** D is a domain such that every simple closed contour within it encloses only points of D.
A **multiply connected domain** is a domain that is not simply connected domain.

Thm (Cauchy-Goursat (C-G) Theorem)
Naive:
If f is analytic at all points interior to and on a simple closed contour C and f' is CTS at all points interior to and on C, then $\oint_C f(z)\ \mathrm{d}z$.
Version 1:
If f is analytic at all points interior to and on a simple closed contour C, then $\oint_C f(z)\ \mathrm{d}z$.
Version 2:
If D is a simply connected domain and f is analytic in D, then $\int_C f(z)\ \mathrm{d}z = 0$ for every closed contour C lying in D.
Version 3:
Suppose that (a) C is simply closed contour (pos. oriented) (b) C_k for $k = 1, \ldots, n$ are simple closed contours interior to C that are disjoint and whose interiors have no commont points (negatively oriented).
If f is analytic on all of these contours and throughout the multiply connected domain consisting of points inside C and exterior to each C_k, then

$$\int_C f(z)\ \mathrm{d}z + \sum_{k=1}^{n} \int_{C_k} f(z)\ \mathrm{d}z = 0$$

Cor of C-G Version 2. f analytic throughout a simply connected domain D, then f must have an anti-derivative on D.
Cor of C-G Version 2. Entire functions always possess anti-derivatives.
Cor of C-G Version 3 (Principle of Deformation of Path): C_1 and C_2 are positively oriented simple closed contours, where C_1 is interior to C_2. Let R be the closed region consisting of these contours and all points between them. If f is analytic on R, then

$$\int_{C_1} f(z)\ \mathrm{d}z = \int_{C_2} f(z)\ \mathrm{d}z$$

Thm (Cauchy Integral Formula)
f analytic everywhere inside and on a simple closed contour C (positively oriented). If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, \mathrm{d}z$$

Rmk: f is analytic in R then values of f interior to C are completely determied by values of f on C.

Thm (Cauchy Integral Formula Extensions)
f analytic inside and on a simple closed contour C (positively oriented). If z_0 is any point interior to C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

Them (Miracle Number 1)
If f is analytic at z_0, then its derivatives of all orders are analytic at z_0.
Cor. $f = u + iv$. If f is analytic at z_0 then u and v have CTS partial derivatives of all orders at $z_0 = (x, y_0)$ (stronger statement than the second condition of Harmonic, which we had put off when we discussed above).

Thm (Morena's Theorem)
Let f be CTS on a domain D. If $\int_C f(z) \, \mathrm{d}z = 0$ for any closed contour C in D then f is analytic in D.

Thm (Cauchy Inequality/Cauchy Estimate)
Let f be analytic inside and on a positively oriented circle C_R centered at z_0 with radius R. If M_R denotees the max value of $|f(z)|$ on C_R, then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n} \qquad (n \geq 1)$$

Thm (Liouville's Theorem; Miracle #2) If f is entire and bounded in $\mathbb{C}$, then $f(z)$ is constant in $\mathbb{C}$.

Thm (Fundamental Theorem of Algebra)
Any polynomial of degree $n \geq 1$, $P(z) = \sum_{k=0}^{n} a_k z^k$ with $a_k \neq 0$ has at least one zero.
Cor.
Every polynomial P of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If these roots are denoted by $w_1, \cdots, w_n$, then

$$P(z) = a_n \prod_{k=1}^{n} (z - w_k)$$

Thm (Maximum Modulus Principle)
If f is analytic and not constant in a domain D, then $|f(z)|$ has no maximum value in D.
Cor. If f CTS in $\overline{D}$ and f is analytic and not constant in D, then $|f(z)|$ reaches max somewhere on the boundary ∂D.

10 Series and Sequences

Def $\{z_n\}_{n=1}^{\infty}$ has a **limit** z if

$$\forall \epsilon > 0 \exists n_0 > 0 : \forall n > n_0 : |z_n - z| < \epsilon$$

Thm

$$\lim_{n\to\infty} z_n = z \iff \begin{cases} \lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} z \\ \lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} z \end{cases}$$

Def

$$\sum_{n=1}^{\infty} z_n = S$$

we say $\sum_{n=1}^{\infty} z_n$ converges to S if $S_N = \sum_{n=1}^{N} z_n$ partial sums satisfy

$$\lim_{N\to\infty} S_N = S$$

Thm

$$\sum_{n=1}^{\infty} z_n = S \iff \begin{cases} \sum_{n=1}^{\infty} \operatorname{Re} z_n = \operatorname{Re} S \\ \sum_{n=1}^{\infty} \operatorname{Im} z_n = \operatorname{Im} S \end{cases}$$

If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

Def.
A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.
Property: Absolute convergence implies convergence.

Geometric Series If $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Uniform convergence $S_n(x) \to S(x)$ uniformly if

$$\forall \epsilon > 0, \exists n_0 > 0 : \forall n > n_0 : \forall x : |S_n(x) - S(x)| < \epsilon$$

If a sequence of CTS functions converges uniformly to a function, then that function is CTS>
Interchange of limits and derivatives/integrals requires uniform convergence.

Thm (Weierstrass M test)
If $\forall n : |a_n(x)| \le M_n \ge 0$ and $\sum_{n=1}^{\infty} M_n$ converges then $\sum_{n=1}^{\infty} a_n(x)$ converges uniformly in x.

10.1 Taylor and Laurent Series

Thm (Taylor Theorem; Miracle #3)
If f is analytic in a disk $D = \{|z - z_0| < R_0\}$, then $f(z)$ has a Taylor series around z_0,

$$\forall z \in D : f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

List of Maclaurin Series W11 Tuesday Lecture Notes.

Thm (Laurent Thm)
If f analytic in a annular domain $D = \{R_1 < |z - z_0| < R_2\}$ and C: any positively oriented simple closed contour around z_0 in D, then for all $z \in D$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \qquad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}}\, dz.$$

Rmk: Alternative phrasing,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \qquad c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

10.2 Power Series

Thm
If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at $z = z_1 \neq z_0$, then it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1 := |z_1 - z_0|$.

Def: largest disk centered at z_0 such that the series converges is called the **disk of convergence** i.e. the open set $D = \{z \in \mathbb{C} : |z - z_0| < R\}$ where R is the **convergence radius**.

Thm: If z_1 is a point inside of disk of convergence D of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, then the series converges uniformly in the closed disk $|z - z_0| \leq R_1 := |z_1 = z_0|$.

In general, we the radius of convergence $R = \frac{1}{\limsup \sqrt[n]{|a_n|}}$ Further, if the limit $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists then $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

Thm (General Result) Consider $0 \leq R \leq \infty$ the convergence radius of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

1. If $|z - z_0| < R$, the series converges absolutely.
2. If $|z - z_0| > R$, the series diverges.
3. For any fixed $r < R$, series converges uniformly for the closed disk $\{z : |z - z_0| \leq r\}$.

Rmk. Laurent Series have inner radius $R_- = \frac{1}{\limsup \sqrt[n]{|b_n|}}$

Thm.
$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ represents a CTS function $S(z)$ at each point insider its disk of convergence $|z - z_0| < R$.

Thm. Integration by terms.
Let $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ on $D = B_R(z_0)$. Let C be any contour in D and $g : C \to \mathbb{C}$ be CTS. Then,

$$\int_C g(z)S(z)\ \mathrm{d}z$$

Cor. $S(z)$ is analytic in D, its disk of convergence.
Cor. $f : D \to \mathbb{C}$ analytic in $D = B_R(z_0)$ if and only if $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ in D.

Thm. (diff by terms)
If $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ w/ conv. disk $D = B_R(z_0)$, then

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

in convergence disk D.

10.2.1 Uniqueness of Taylor/Laurent Series

$$\left(\forall z \in B_R(z_0) : \sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n\right) \implies (\forall n : a_n = b_n)$$

$$\left(\forall z : R_1 < |z-z_0| < R_2 \implies \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n\right) \implies (\forall n : a_n = b_n)$$

10.3 Algebra of Power Series

Consider $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converging on a disk $D_1 = \{|z-z_0| < R_1\}$ and $g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converging on a disk $D_2 = \{|z-z_0| < R_2\}$.
The following are true:

- f, g are analytic on D_1, D_2 respectively.
- fg is analytic on $D_3 = \{|z-z_0| < \min\{R_1, R_2\}\}$
- fg has a Taylor Series on D_3 of the form $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

11 Residues and Poles

Def. A singular point z_0 is an **Isolated Singular Point** of f if there exists some deleted neighborhood $B'_\epsilon(z_0)$ in which f is analytic.

Def. The residue of $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$, where this Laurent series holds on $B'_\epsilon(z_0)$, at z_0 is given by

$$\text{Res}_{z=z_0} f(z) = b_1$$

Thm (Residue Theorem) Let C be a simple positively-oriented closed contour. Let f be analytic inside and on C except for a fininite number of isolated singular points z_k for $k = 1, \cdots, n$ inside C.

$$\int_C f(z)\ \mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_{z=z_k} f(z)$$

Def. f is analytic on $R_1 < |z| < \infty$, then we call ∞ an isolated singular point of f. For C_0 a *positively* oriented curve $|z| = R_0 > R_1$, then

$$\mathrm{Res}_{z=\infty} f(z) := -\frac{1}{2\pi i}\int_{C_0} f(z)\ \mathrm{d}z = -\mathrm{Res}_{z=0}\left(\frac{1}{z^2} f\left(z^{-1}\right)\right)$$

Let f be analytic on $\mathbb{C} \setminus \{z_1, \dots, z_n\}$ such that $\{z_k\}_{k=1}^{n}$ is a finite set of isolated singular points interior to a simple closed contour C, then

$$\int_C f(z)\ \mathrm{d}z = 2\pi i\,\mathrm{Res}_{z=0}\left(\frac{1}{z^2} f\left(z^{-1}\right)\right)$$

Def. Types of Isolated Singular Points of a function f, such that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ on $B'_{R_2}(z_0)$.

- **Removable**: $\forall n > 0 : b_n = 0$.
- **Essential**: For an infinite number of $n > 0$, $b_n \neq 0$.
- **Pole of order** m: $b_m \neq 0$ and $\forall n > m : b_n = 0$.

Thm. If z_0 is an isolated singular point of f, then the following are equivalent:

1. z_0 is a pole of order $m > 0$ of f
2. $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0.

Further,

$$\mathrm{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Def. z_0 is a zero of order m if $0 = f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0$ and $f^{(m)}(z_0) \neq 0$.

Thm.
If f is analytic at z_0 the following are equivalent:

1. z_0 is a pole of order $m > 0$ of f
2. $f(z) = (z - z_0)^m g(z)$ where $g(z)$ is analytic and non-zero at z_0.

Thm. Let p, q be two functions that are analytic at z_0 w/ $p(z_0) \neq 0$, and q have a zero of order m at z_0. Then $\frac{p(z)}{q(z)}$ has a pole of order m at z_0.

Thm. p, q analytic at z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$, then z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and $\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$.

Thm. Let f be analytic at z_0. If $f(z_0) = 0$, then either $f \equiv 0$ in some neighborhood of z_0 or there exists some deleted neighborhood $B'_\epsilon(z_0)$ such that $\forall z \in B'_\epsilon(z_0) : f(z) \neq 0$.

Miracle #4 (Analytic Continuation)
Thm. A function analytic in a domain D is uniquely determined over D by its values on a (smaller) domain or along a line segment contained in D.
Alternative Phrasing: Suppose f, g analytic in D. If $f = g$ on a smaller domain/line segment contained in D, then $f \equiv g$ in D.

Lemma. Let f be analytic in a domain D. If $f = 0$ at each point of a domain/line segment contained in D, then $f \equiv 0$ in D.

Thm. Reflection Principle.
f analytic in some domain D that contains a segment of x-axis and whose lower half is symmetric to upper half wrt the x axis. If $f(x)$ is real for each x on this segment, then

$$\overline{f(z)} = f(\overline{z}).$$

Thm. If z_0 is a removable singularity of a function f, then f is bounded and analytic in some deleted neighborhood, $B'_\epsilon(z_0)$.

Thm (Riemann's Theorem of Removable Singularities). If f is bounded and analytic in some $B'_\epsilon(z_0)$ and f is not analytic at z_0, then z_0 is a removable singularity of f.

If z_0 is a pole of f, then $\lim_{z \to z_0} f(z) = \infty$.

Thm (Casorati-Weierstrass Theorem). If z_0 is an essential singularity of f (an analytic function in $B'_{\delta_0}(z_0)$), then

$$\forall w_0 \in \mathbb{C} : \forall \epsilon > 0 : \forall 0 < \delta < \delta_0 : \exists z \in B'_\delta(z_0) : |f(z) - w_0| < \epsilon$$

Intuitive Statement of the Great Picard Theorem: In each neighborhood of an essential singularity, the function takes values of every complex number infinitely many times, w/ one possible exception.

12 Computing Indefinite Integrals

Def.

$$P.V. \int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = \lim_{R \to \infty} \int_{-R}^{R} f(x)\ \mathrm{d}x$$

Rmk.
If f is CTS on $(-\infty, \infty)$ and odd, then the PV is 0.
If f is CTS on $(-\infty, \infty)$ and even, then the PV is $PV \int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = 2\int_0^{\infty} f(x)\ \mathrm{d}x$.

Form 1: $\frac{p(x)}{q(x)}, x \in \mathbb{R}$ rational functions and even. w/ p, q not having common factor.
1. Find all singularities (poles) in the upper half-plane.
2. Compute the residue and apply residue theorem.
3. Use ML estimate to show integral along C_R goes to 0.
This holds for:

$$\deg Q \geq \deg P + 2$$

Form 2: $\frac{P(x)}{Q(x)} \sin(ax)$, $\frac{P(x)}{Q(x)} \cos(ax)$, for $a > 0$.
Trick analyze

$$f(z) = \frac{P(z)}{Q(z)} e^{iaz}$$

Jordan Inequality.

$$\forall R > 0 : \int_0^{\pi} e^{-R\sin\theta} \, \mathrm{d}\theta < \frac{\pi}{R}$$

Jordan Lemma.
Let $g(z) < M_R$ on $C_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$, then

$$\left|\int_{C_R} g(z)e^{iaz} \, \mathrm{d}z\right| < \frac{M_R \pi}{a}$$

Form 3:
$\int_{-\infty}^{+\infty} f(x) \, \mathrm{d}x$, where f has a pole in $\mathbb{R}$.
Add a negatively-oriented loop that hops around the pole on $\mathbb{R}$.

Form 4:
$\int_{-\infty}^{+\infty} f(x) \, \mathrm{d}x$, where f involves $\log(x)$, x^a or another function that involves branches.
Pick a branch along the negative imaginary axis, and add a negatively-oriented loop that hops around the pole at the origin.