Thereom List for Math 123 (ODE) w/ Di Fang.

Parth Nobel

Basic Defs

ODE:

$$\begin{cases} f : D \subset \mathbb{R}^{n+1} \mapsto \mathbb{R} \\ y^{(n)} = f(t, y(t), y'(t), \cdots, y^{(n-1)}(t)) \end{cases}$$

Solution of a Diff Eq:

$\phi(t)$ solves an ODE on $I = (t_1, t_2)$ if

1. $\phi(t), \phi'(t), \cdots, \phi^{(n-1)}(t), \phi^{(n)}(t)$ exists for $t \in I$
2. $(\phi(t), \phi'(t), \cdots, \phi^{(n-1)}(t)) \in D$ for $t \in I$
3. $\phi^{(n)}(t) = f(t, \phi(t), \phi'(t), \cdots, \phi^{(n-1)}(t))$

Solution Techniques

Integrating Factors

$$\dot{y}(t) + a(t)y(t) = b(t)$$

Giving $m(t) \triangleq e^{\int a(t)\,\mathrm{d}t}$

$$y(t) = \frac{1}{m(t)}\left[\int m(t)b(t)\,\mathrm{d}t + C\right]$$

Bernoulli Eq.

$$\frac{dy}{dt} + a(t)y = b(t)y^n \qquad n \geq 0$$

Substitute $z = y^{1-n} \implies \frac{1}{1-n}z' = y^{-n}\frac{dy}{dt}$

2nd Order ORDE (linear homo)

$$\ddot{y} + a(t)\dot{y} + b(t)y = 0$$

If y_1 and y_2 are solns, then so is any linear combination.

Theorem: Two solns $y_1(t)$ and $y_2(t)$ are linearly dependent iff $W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = 0$.

Existence

Thm: Picard's Existence Theorem

Suppose f defined on a rectangle R of size $2a \times 2b$ is bounded, i.e. $|f(t,y)| \leq M \quad \forall (t,y) \in R. \quad M > 0.$ and is a cts function satisfying Lipschitz condition

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$$

for some constant $L > 0$.

Then the IVP has a soln on the interval $\{t : |t - t_0| \leq \alpha\}$ for some constant $\alpha > 0$, $\alpha = \min\{a, \frac{b}{M}\}$.

Picard's Iteration:

$$y_0(s) \triangleq y_0$$

$$y_n(t) \triangleq y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))\,\mathrm{d}s$$

$y(t) = \lim_{n\to\infty} y_n(t)$ exists and solves IVP.

Uniform Convergence (allows interchange of limits and integrals) (Note N before t in the qualifiers)

$$\forall \epsilon > 0, \exists N : \forall n > N, \forall t \in I : |f_n(t) - f(t)| < \epsilon$$

If $|f_n(t)| \leq M_n$ for all $t \in I$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(t)$ converges uniformly.

Peano's Existence Theorem

Suppose f is CTS on rectangle R. Then there exists a soln of IVP on the interval $|t - t_0| < \alpha$ for some $\alpha > 0$

Uniqueness

Thm (Gronwall's Ineq.)

Let $K \geq 0$ constant, f and g are cts *non-negative* functions defined on $t \in [a, b]$ satisfying

$$\forall t \in [a, b] : f(t) \leq k + \int_a^t f(s)g(s)\,\mathrm{d}s$$

$$f(t) \leq k \exp\left(\int_a^t g(s)\,\mathrm{d}s\right)$$

Uniqueness Theorem

Suppose f is CTS satisfying Lip. condition, i.e.

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

such that $L > 0$ constant, on the "box" $R = \{(t, y) : |t - t_0| \leq a, |y - y_0| \leq b\}$ then the soln (defined by local existence thm) is unique.

Sufficient condition for Lip

$$\left|\frac{\partial f}{\partial y}\right| \leq L.$$

Global Existence

Lemma

Suppose f is CTS in a domain D, $|f| \leq M$ in D. Let ϕ be a soln of $\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \\ y(t_0) = y_0 \end{cases}$ that exists a finite interval (a, b). Then $\lim_{t \to a^+} \phi(t)$ and $\lim_{t \to b^-} \phi(t)$ exists.

Suppose f is CTS in a given region D satisfying Lip condition.

f is bounded in D. Let $(t_0, y_0) \in D$. Then the unique soln of $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$, passing through the point (t_0, y_0) can be extended until its graph meets the boundary of D.

Corrollary: If D is (t, y) space, and if f is CTS and Lip on D, then the soln of IVP can be extended uniquely in both directions as long as $|\phi(t)|$ remain finite.

Def: Apriori estimate: $|\phi(t)| \leq M$

Corollary: Consider autonomous system $\begin{cases} y' = f(y) \\ y(t_0) = y \end{cases}$ with $f : \mathbb{R} \to \mathbb{R}$ CTS.

If the solution $\phi(t)$ satisfies $|\phi(t)| \leq M$ wherever $\phi(t)$ exists then $I = (-\infty, \infty)$ which gives global existence of solution.

Thm: f CTS in (t, y), bdd, lip in y. in D Lip. const: L.

Let ϕ be the soln of the IVP with $y(t_0) = y_0$, and ψ be the soln of IVP with $y(t_0) = \tilde{y}_0$

Suppose ϕ, ψ exist on some interval $a < t < b$.

Then $\forall \epsilon > 0, \exists \delta > 0 : |y_0 - \tilde{y}_0| < \delta \implies (\forall t \in (a, b) : |\phi(t) - \psi(t)| < \epsilon)$

Thm: Let f and g def on D. CTS in (t, y), bdd $\begin{cases} |f| \leq M \\ |g| \leq M \end{cases}$

Lip cts y. w/ same Lip constant L.

Let ϕ be $\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$ and ψ be $\begin{cases} y' = g(t, y) \\ y(t_0) = y_0 \end{cases}$ exists a common interval $a < t < b$. Suppose $|f(t, y) - g(t, y)| \leq \epsilon \quad \forall (t, y) \in D$. Then solns ϕ and ψ satisfy the estimate $|\phi(t) - \psi(t)| \leq \epsilon(b - a)\exp(L|t - t_0|)$.

Linear Systems

Thm: $\frac{\mathrm{d}y}{\mathrm{d}t} = A(t)y + g(t)$ with $y(t_0) = y_0$. If $A(t)$, $g(t)$ are CTS on some interval $[a, b]$ and $t_0 \in [a, b], y_0 < \infty$ then the system has a unique soln $\phi(t)$ satisfying $\phi(t_0) = y_0$ and existing on $[a, b]$.

Thm $\frac{\mathrm{d}y}{\mathrm{d}t} = A(t)y$ with $y \in \mathbb{R}^n$ (W5B)

If $n \times n$ complex $A(t)$ is CTS on an interval I, then the soln of the system on I form a vector space of dimension n over complex numbers.

Def. Linearliy indep. solns $\phi_1, \cdots, \phi_n$ are called fundamental set of solns.

$$\Phi = \begin{bmatrix} \phi_1 & \cdots & \phi_n \end{bmatrix}$$

1. Satisfies $\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A(t)\Phi$
2. $\forall \vec{c} \in \mathbb{C}^n : \Phi(t)\vec{c}$ solves IVP.
3. $\forall \psi(t) \in S : \exists \vec{c} : \psi(t) = \Phi(t)\vec{c}$
4. $\forall t : \det(\Phi(t)) \neq 0$

Lemma: $\Phi(t)$ satisfies IVP on an interval I, it is a fund. matrix of IVP on I iff $\forall t \in I : \det(\Phi(t)) \neq 0$

Thm: Abel's Formula

If Φ is a fund. matrix of IVP on I, and $t_0 \in I$, then

$$\det \Phi(t) = \det \Phi(t_0) \exp\left(\int_{t_0}^{t} \sum_{k=1}^{n} A_{kk}(s)\,\mathrm{d}s \right)$$

A soln. matrix $\Phi(t)$ of IVP is a fund. matrix iff $\det(\Phi(t)) \neq 0$ for some $t = t_0$.

Cor: $\Phi(t)$ is a fund. matrix of IVP on I and C is a non-singular const matrix, then $\Phi(t)$C is a fund. matrix of IVP on I.

Variation of const formula

$$y(t) = \Phi(t)\Phi^{-1}(t_0)y_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)\mathrm{d}s$$

Matrix exponential:

$$e^M \triangleq \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

Properties:

1. $e^{\mathbf{0}} = I$
2. If $AB = BA$ then $e^{A+B} = e^A e^B$ and $Ae^B = e^B A$
3. e^A is always invertible.
4. If T is nonsingular $n \times n$ mmatrix, then $e^{TAT^{-1}} = Te^A T^{-1}$

The Matrix $\Phi(t) = e^{At}$ is a fund. matrix of $\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A\Phi(t)$ w/ $\Phi(0) = I$

Thm: λ is a complex e-val of real matrix A w/ e-vec v then $\bar{\lambda}$ is also an e-val w/ e-vec $\bar{\vec{v}}$

See Lecture 7A for the construction of the existence of V for all A with distinct e-vals such that $AV = VD$ where D is not quite diagonal, but still easy to compute a fundamental matrix of.

Def: For a given e-val λ, vector v is called a **generalized eigenvector** of rank (or index) r if

$$(A - \lambda I)^r v = 0 \wedge (A - \lambda I)^{r-1} v \neq 0$$

Def: **Chain of generalized eigenvectors** given a generalized eigenvector v of rank r, is given by $v_r = v$, and

$$v_{r-i} = (A - \lambda I)^i v = (A - \lambda I) v_{r-i+1}.$$

Lemma: gen e-vecs in a chain are linearly independent.

Theorem: Given a chain of gen e-vecs of length r w/ e-vals λ we define for $k \in 1, 2, \ldots, r$,

$$y_k(t) = e^{\lambda t} \sum_{j=1}^{k} \frac{t^{r-i}}{(r-i)!} v_i$$

which forms r independent solutions of $\frac{dy}{dt} = Ay$

Lemma

If $\lambda_1, \cdots, \lambda_k$ are the distinct e-vals of A, where λ_j has multiplicity n_j and $n_1 + \cdots + n_k = n$. Then $\forall \rho > \max_{i \leq j \leq k} \operatorname{Re}\{\lambda_j\} \exists K > 0 : |e^{tA}| \leq K e^{\rho t}$.

Remark $\forall \rho \geq \max_{i \leq j \leq k} \operatorname{Re}\{\lambda_j\} \exists K > 0 : |e^{tA}| \leq K e^{\rho t}$ iff **all** e-vals with $\max_j \operatorname{Re}\{\lambda_j\}$ are simple, in the geometric multiplicity = algebraic multiplicity.

Cor: If all e-vals of A have real parts negative, then every solution $\phi(t)$ of $\frac{dy}{dt} = Ay$ approaches 0 as $t \to \infty$

Suppose that in the non-homo linear system $\frac{dy}{dt} = Ay + g(t)$ the function $g(t)$ grow no faster than an exponential function, that is $\exists a \in \mathbb{R}, M > 0, T > 0 : t \geq T \implies |g(t)| \leq e^{at}$. Then every solution ϕ of the system grows no faster than an exponential function, that is,

$$\exists K > 0, T > 0, b \in \mathbb{R} : t \geq T \implies |\phi(t)| \leq Ke^{bt}$$

Remarks:

1. $\phi'(t) \leq \tilde{C}e^{\max\{a,b\}t}$.
2. b can be picked as $\max\{a, \rho\}$, where $\rho > \max_j\{\mathrm{Re}\{\lambda_j\}\}$

Cor:

If $\mathrm{Re}\{\lambda_j\} < 0$ for all j and $a < 0$, then

$$\begin{cases} \lim_{t\to\infty} y(t) = 0 \\ \lim_{t\to\infty} y'(t) = 0 \end{cases}$$

See Lecture 10A and 10B for phase portraits.

Linear Periodic Time-Varying ODEs (LPTV ODE)

Floquet Theorem Let $A(t) \in \mathbb{R}^{n\times n}$ CTS periodic matrix with period T. Let $\Phi(t)$ be a fundamental matrix of

$$\dot{y} = A(t)y \tag{LPTV}$$

.

Then there exists a periodic nonsingular matrix $P(t)$ with period T and a constant matrix R s.t.

$$\Phi(t) = P(t)e^{tR}$$

Remarks:

1. There exists $Q(t)$ real and periodic and S a real constant such that $\Phi(t) = Q(t)e^{tS}$
2. For all $y(t)$ that solves (LPTV), $y(t) = P(t)u(t)$ such that $\frac{du}{dt} = Ru$.

Cor 1.

There exists a non-zero solution of (LPTV) $y(t)$ such that $y(t+T) = \lambda y(t)$ iff λ is an eval of e^{TR}.

Def. The evals of $C = e^{TR}$ are called **Floquet Multiplier** and denoted λ_i.

Def. The evals of R are called **Floquet Exponents** or **Characteristic exponent** and denoted ρ_i.

Note that there is not a one-to-one correspondence of λ to ρ.

Cor 2.

If Floquet Exponents of (LPTV) have negative real parts (or equivalently if multipliers have magnitude strictly less than 1), then all solutions of (LPTV) approach zero as $t \to \infty$.

Thm

Let $A(t)$ be a matrix with period T and $g(t)$ be periodic with same period T. Consider the perturbed system

$$\dot{y} = A(t)y + g(t) \tag{PLPTV}$$

.

A solution $y(t)$ of (PLPTV) is periodic of period T in t iff the soln satisfies $y(T) = y(0)$.

Thm

(PPTV) has periodic solution of period T for **any** periodic forcing vector g of period T iff $y' = A(t)y$ has no periodic solution of period T except trivial solutions.

Lyapunov Stability

Def

1. $\frac{dy}{dt} = f(t, y)$ denote $\phi(t)$ as a solution w/ IC $\phi(t_0) = \phi_0$. $\phi(t)$ is said to be **stable** if $\forall \epsilon > 0, \exists \delta > 0 : \|\phi(t_0) - y_0\|_2 < \delta$, the solution $y(t)$ of the solution passing through (t_0, y_0) satisfies $\|\phi(t) - y(t)\| < \epsilon$ for $t \geq t_0$.
2. **Asymptotic stable** if it is stable and $\exists \delta_0 > 0$ such that whenever $\|(t_0) - y_0\|_2 < \delta_0$, $\lim_{t \to \infty} \|y(t) - \phi(t)\|_2 = 0$.
3. **unstable** if it is not stable.

Lemma.

The stability of a solution to $\frac{dy}{dt} = Ay$ is equivalent to the stability of the zero solution $y(t) \equiv 0$.

Thm.

$$\frac{dy}{dt} = Ay$$

a) If all e-vals have negative real part, $y \equiv 0$ is asymptotically stable.
b) If all e-vals have non-positive real part, and e-vals with zzero real part are simple, then $y \equiv 0$ is stable.
c) If exists an e-val with positive real part or a non-simple e-vals with zero real part, then $y \equiv 0$ is unstable.

Theorem (periodic)

$y' = A(t)y$ with $A(t)$ is periodic with period T

a) If modulus of multiplier all < 1, zero soln is asymptotically stable.
b) If modulus of multiplier all < 1 or $= 1$, zero soln is stable.
c) If exists a multplier with modulus > 1, zero soln is unstable.

Theorem

For $y' = (A(t) + B(t))y$

Let all evals of A have real part negative and $B(t)$ CTS for $0 \leq t < \infty$ w/ $\lim_{t\to\infty} B(t) = 0$.

Then the zero solution is globally asymptotically stable.

Cor.

$|e^{At}| \leq Ke^{-\rho t}$ for some $K > 0, \rho > 0$ for all $t \geq 0$. Let $B(t)$ be CTS for $t \geq 0$ and $\exists T > 0$ s.t. $t \geq T \implies |B(t)| \leq \frac{\sigma}{K}$. Then zero solution is globally asymptotically stable.

Linearization

$$y' = F(y) \qquad \text{(ANLE)}$$

. $y = y^*$ is an equilibrium solution if $F(y^*) = 0$. The function $z(t) \triangleq y(t) - y^*$ satisfies $\frac{dz}{dt} = F(y^*) + D_yF(y^*)z + g(z) = Az + g(z)$ with $A = D_yF(y^*)$ is the Jacobian of F with respect to y^* and with $g(z)$ CTS, having a fixed point at 0 i.e. $g(0) = 0$, and satisfying $\lim_{z\to 0} \frac{|g(z)|}{|z|} = 0$. This is the linearization of F at y^*.

Thm

Consider "almost linear" system. $y' = Ay + f(t, y)$. Suppose all e-vals of A have negative real parts. $f(t, y)$ CTS in (t, y) for $0 < t < \infty$, $|y| < \tilde{K}$ where $\tilde{K} > 0$ is a constant, and f is small in the sense that $\lim_{y\to 0} \frac{|f(t,y)|}{|y|} = 0$ uniformly in t on $0 \le t < \infty$. Then the solution $y \equiv 0$ is asymptotically stable.

For Bootstrapping arguments, see HW9 Q3 for an example proof or Lecture 12B.

Def

If $A = Df(y^*)$ has no e-val w/ zero real part, we call y^* a **hyperbolic** equil. solution.

Notation

$\phi_t(y)$ is the solution to a given diff eq with initial condition y evaluated at time t.

Theorem (Hartman-Groan Theorem)

Informally: "In hypperbolic cases, the behaviour of solutions near equilibria of a nonlinear system is qualitatively the same as its linearization."

Formally:

Let y^* be an equilibrium solution of $y' = f(y)$, f is CTS and CTSly differentiable.

Assume that the linearization matrix at y^* ($A = Df(y^*)$ has no -e-val with zero real part (it is hyperbolic).

There there exists a neighborhood U of y^* such taht "the behaviour of solutions of $y' = f(y)$ in U is qualitatively the same as its linearization." Formally, there exists a CTS bijection H w/ continuous inverse (homeomorphism), w/ domain U such that for any $y_0 \in U$, $H \circ \phi_t(y_0) = e^{tA} H(y_0)$.

Lyapunov Second Method

Consider $\frac{d}{dt}\vec{y} = \vec{f}(\vec{y})$ with $\vec{y} \in \mathbb{R}^n$.

If there exists a function $V : \mathbb{R}^n - > \mathbb{R}$ ctsly diff on some neighborhood Ω containing the origin and 1. V is pos. def. i.e. $V(0) = 0$ and $\forall \vec{y} \in \Omega \setminus \{0\}$: $V(\vec{y}) > 0$. 1. $V^*(\vec{y}) = V(\vec{y}) \cdot \vec{f}(\vec{y}) \leq 0$ on Ω.

Then zero solution of $\dot{\vec{y}} = \vec{f}(\vec{y})$ is stable.

Consider $\frac{d}{dt}\vec{y} = \vec{f}(\vec{y})$ with $\vec{y} \in \mathbb{R}^n$.

If there exists a function $V : \mathbb{R}^n - > \mathbb{R}$ ctsly diff on some neighborhood Ω containing the origin and 1. V is pos. def. i.e. $V(0) = 0$ and $\forall \vec{y} \in \Omega \setminus \{0\}$: $V(\vec{y}) > 0$. 1. V^* is neg.def i.e. $V(0) = 0$ $\forall \vec{y} \in \Omega \setminus \{0\} : V^*(\vec{y}) < 0$

Then zero solution of $\dot{\vec{y}} = \vec{f}(\vec{y})$ is asymptotically stable.

If $|y| \to \infty$, $V(y) \to \infty$, then the zero solution is globally asymptotically stable.

Notation

$\psi(t; t_0, y_0)$ the flow of an autonomous system, solves $y' = f(y)$ with $y(t_0) = y_0$ by definition. If $t_0 = 0$ it can be denoted as $\psi_t(y_0) = \psi(t; y_0) = \psi(t; 0, y_0)$ For an autonomous system it satisfies,

1. $\psi(t; t_0, y_0) = \psi(t - t_0; 0, y_0)$
2. $\psi_{t+s} = \psi_t \circ \psi_s$ i.e. $\psi_{t+s}(y_0) = \psi_t(\psi_s(y_0))$

Def

P is an **invariant set** of an auto. system if $\forall y_0 \in P, \forall t \geq 0 : \phi_t(y_0) \in P$.

Def

The **Positive semi-orbital/Negative semi-orbital** of a solution is its behaviour on $t \geq 0/t \leq 0$

Thm (La Salle's Invariance Principle)

If V is a Lyapunov function on Ω and ctsly diffable $E = \{y \in \Omega : V^*(y) = 0\}$ with M the largest invariant set in E.

Consider a solution $\phi_t(y_0)$ that is bounded whose positive semi-orbital lies in Ω for $t \geq 0$, then dis$(\psi_t(y_0, M) \to 0$ as $t \to \infty$

Cor

If $V(y) \to \infty$ as $|y| \to \infty$ and $V^* \leq 0$ on $\mathbb{R}^n$, then every solution $y' = f(y)$ is bounded and approaches M.

In particular if $M = \{0\}$ then the system is globally asymptotically stable.

Thm

$U(x)$ is a potential function, consider $x'' + U'(x) = 0$ then

1) equilibrium points are the critical points of $U(x)$
2) strict local maximum of $U(x)$ is a saddle
3) strict local minimum of $U(x)$ is a center