EE120 Notes

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1 Fourier Transformations

Synthesis/Analysis Eqns

Note for intuition: we derived the DTFS and CTFS analysis equations by projecting our time-domain signal on the k^{th} basis vector ($e^{ik\omega_0 n}$ or $e^{ik\omega_0 t}$)

(a) DTFS (discrete-time, periodic):

$$x(n) = \sum_{k \in \langle p \rangle} X(k) e^{i\omega_0 kn}$$

$$X(k) = \frac{1}{p} \sum_{n \in \langle p \rangle} X(n) e^{-i\omega_0 kn}$$

Aside: the DFT is similar to the DTFS except the is a $1/p$ term in the synthesis equation and not the analysis equation.

(b) DTFT (discrete-time, general):

$$x(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega) e^{i\omega n} d\omega$$

$$X(\omega) = \sum_{n \in \mathbb{Z}} x(n) e^{-i\omega n}$$

Note: in order to use the analysis equation, the signal has to be absolutely summable. If the signal is not absolutely summable but instead square summable (finite energy), the signal still has a DTFT but you have to use the synthesis equation and pattern-match. This also applies to the CTFT.

(c) CTFS (continuous-time, periodic):

$$x(t) = \sum_{k \in \mathbb{Z}} X(k) e^{ik\omega_0 t}$$

$$X(k) = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-i\omega_0 kt} dt$$

(d) CTFT (continuous-time, general):

$$x(t) = \frac{1}{2\pi} \int_{\mathbb{R}} X(\omega) e^{i\omega t} d\omega$$

$$X(\omega) = \int_{\mathbb{R}} x(t) e^{-i\omega t} dt$$

Properties of CTFT/DTFT

Note: Let $\mathcal{F}\{x(t)\}(\cdot)$ denote the CTFT of $x(\cdot)$

(a) (DTFT, CTFT) Rayleigh-Plancherel-Parseval Identity

$$\langle x \mid y \rangle = \frac{1}{2\pi}\langle X \mid Y \rangle$$

(b) (DTFT, CTFT) Time shift

$$\mathcal{F}\{x(t-T)\}(\omega) = e^{-i\omega T}\,\mathcal{F}\{x(t)\}(\omega)$$

(c) (CTFT) Time scale

$$\mathcal{F}\{x(at)\}(\omega) = \frac{1}{|a|}\,\mathcal{F}\{x(t)\}\left(\frac{\omega}{a}\right)$$

(d) (DTFT, CTFT) Conjugate symmetry (for real signals):

$$X^*(\omega) = X(-\omega)$$

The DTFT/CTFT of a real and even signal will also be real and even. The DTFT/CTFT of a real and odd signal will be imaginary and odd.

(e) (CTFT) Differentiation in time

$$\mathcal{F}\left\{\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right\}(\omega) = i\omega\,\mathcal{F}\{x(t)\}(\omega)$$

(f) (CTFT) Differentiation in frequency

$$\mathcal{F}\{tx(t)\}(\omega) = i\frac{\mathrm{d}\,\mathcal{F}\{x(t)\}}{\mathrm{d}\omega}(\omega)$$

(g) (DTFT, CTFT) Convolution property

$$\mathcal{F}\{x(t) * y(t)\}(\omega) = \mathcal{F}\{x(t)\}(\omega)\,\mathcal{F}\{y(t)\}(\omega)$$

(h) (DTFT, CTFT) Modulation property

$$\mathcal{F}\{x(t)y(t)\}(\omega) = \frac{1}{2\pi}\,\mathcal{F}\{x(t)\}(\omega) * \mathcal{F}\{y(t)\}(\omega)$$

Note: this also applies to the DTFT, but the convolution is a circular convolution (over a 2π range). To perform circular convolution, keep the more complicated signal in place, only keep one 2π cycle of the other, and perform regular convolution. Also, like shown for the CTFT, divide by 2π.

(i) Iterated CTFT

$$\mathcal{F}\{\,\mathcal{F}\{x(\tau)\}(t)\}(\omega) = 2\pi x(-\omega)$$

Fourier Transforms of Common Signals

(a) CTFT of a delta

$$\mathcal{F}\left\{\delta(t-T)\right\}(\omega) = e^{-i\omega T}$$

(b) CTFT of a complex exponential

$$\mathcal{F}\left\{e^{i\omega_c t}\right\}(\omega) = 2\pi\delta(\omega-\omega_c)$$

(c) CTFT of a constant

$$\mathcal{F}\left\{1\right\}(\omega) = 2\pi\delta(\omega)$$

(d) CTFT of the unit step

$$\mathcal{F}\left\{u(t)\right\}(\omega) = \frac{1}{i\omega} + \pi\delta(\omega)$$

(e) CTFS of ideal LPF (width T, height $1/T$)

$$X_k = \frac{1}{\pi k t}\sin(k\omega_0 T/2)$$

(f) DTFT of ideal LPF (width $2B$, height A)

$$\frac{A}{\pi n}\sin(Bn)$$

2 Amplitude Modulation

Modulation

Multiply your signal $x(t)$ with a carrier signal $c(t) = \cos(\omega_0 t)$. Assume that $X(\omega)$ is band-limited such that, if the bandwidth is $2B$, $B < |\omega_0|$. By the modulation property of the CTFT, the resulting signal in the frequency domain will have two copies of $X(\omega)$, one centered around ω_0 and the other centered around $-\omega_0$ and both scaled by $1/2$.

Demodulation

To demodulate, multiply your incoming signal $y(t)$ (assume that there was no corruption in transmission) by $\cos(\omega_0 t)$. The transform of the resulting signal will have a copy of $X(\omega)$ centered at 0 and scaled by $1/2$, and copies at $-2\omega_0$ and $2\omega_0$, both scaled by $1/4$. To recover the original signal, we pass this through a LPF with a gain of 2.

Potential Problems with Demodulation

(a) Phase drift: during demodulation, the signal is instead multiplied by $\cos(\omega_0 t + \theta)$.

$$\hat{x}(t) = \cos(\theta)\, x(t)$$

Depending on the value of θ, the signal will be scaled down, or even zeroed out (at $\pi/2$ or $3\pi/2$). You can deal with phase drift by using an Asynchronous Demodulation circuit consisting of a diode followed by a resistor and capacitor in parallel (measure the voltage across the resistor).

(b) Frequency drift: during demodulation, the signal is instead multiplied by $\cos((\omega_0 + \Delta\omega)t)$.

$$\hat{x}(t) = \cos(\Delta\omega t)\, x(t)$$

You can deal with frequency drift by demodulating $y(t)$ in two parts: one where $y(t)$ is multiplied by $\cos((\omega_0 + \Delta\omega)t)$ and one where $y(t)$ is multiplied by $\sin((\omega_0 + \Delta\omega)t)$. Pass both demodulated signals through a LPF to get:

$$q_1(t) = \cos(\Delta\omega t)\, x(t)$$

$$q_2(t) = \sin(\Delta\omega t)\, x(t)$$

You can recover a **non-negative** signal $x(t)$ as follows:

$$x(t) = \sqrt{q_1(t)^2 + q_2(t)^2} = \sqrt{(\cos^2(\Delta\omega t) + \sin^2(\Delta\omega t))x^2(t)} = |x(t)|$$

3 Sampling Theory

Sampling a CT Signal

Say you have a CT signal $x(t)$ with band-limited ($|\omega| \leq B$) transform $X(\omega)$ that we sample using *sampling period* T_s and *sampling frequency* $\omega_s = \frac{2\pi}{T_s}$.
First, modulate your signal with the impulse train $p(t)$ and convert Dirac deltas to Kronecker deltas:

$$p(t) = \sum_{l=-\infty}^{\infty} \delta(t - lT_s)$$

$$P(\omega) = \frac{2\pi}{T_s} \sum_k \delta(\omega - k\omega_s)$$

The resulting signal $X_p(\omega)$ will have copies of $X_p(\omega)$ centered at integer multiples of ω_s and scaled by $1/T_s$. To recover the signal, you can pass it through a LPF with cutoff $\omega_s/2$. In the time domain, this is represented by sinc interpolation:

$$h(t) = \frac{T_s}{\pi t} \sin(\omega_s/2t) = sinc(t/T_s)$$

Nyquist Rate

To reconstruct a signal with max frequency $\omega = B$ we need to sample at a rate ω_c such that $2B < \omega_c$. Otherwise, there will be **aliasing**, where higher frequencies roll over into lower frequencies. You can pre-process as signal using *anti-alias filtering* before the modulation step to avoid aliasing (by cutting off higher frequencies completely with a LPF).

4 Z-Transform

$$\mathcal{Z}\{x(n)\}(z) = \hat{X}(z) = \sum_{n \in \mathbb{Z}} x(n)\, z^{-n}$$

Note that the $\mathcal{Z}$-Transform doesn't converge for lots z. The region for which it converges is known as the Region of Convergence (RoC).

If a signal is causal, its RoC extends from outermost pole out to infinity.

If a signal is anti-causal, its RoC extends from innermost pole towards the 0.

If a signal is two-sided, the RoC will be between two poles.

If the unit circle is inside the RoC, then the system is BIBO stable.

Properties of the Z-Transform

Time Delay:

$$\mathcal{Z}\{x(n-N)\}(z) = z^{-N}\hat{X}(z)$$

Convolution in time is multiplication in frequency.

$$\mathcal{Z}\{x(n) * y(n)\}(z) = \hat{X}(z)\hat{Y}(z)$$

DTFT is the $\mathcal{Z}$-transform evaluated on the unit circle.

$$\mathcal{F}\left\{x(n)\right\}(\omega) = \mathcal{Z}\{x(n)\}(e^{j\omega})$$

Initial value theorem (causal systems)

$$x(0) = \lim_{z\to\infty} \hat{X}(z)$$

Z-Transform of a LCCDE: For the LCCDE defined by

$$a_0 y(n) + a_1 y(n-1) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$$

the Z-transform $\hat{H}(z)$ is represented by

$$\hat{H}(z) = \frac{\hat{Y}(z)}{\hat{X}(z)} = \frac{b_0 + b_1 z^{-1} + \cdots b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$